Journal of the Egyptian Mathematical Society (2013) **21**, 184–189

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Certain subclasses of uniformly convex functions of order α and type β with varying arguments

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Received 31 December 2012; accepted 16 February 2013
Available online 31 March 2013

KEYWORDS

Univalent functions;
Convex functions;
Starlike functions;
Uniformly convex functions;
Uniformly starlike functions

Abstract In this paper, we define a new subclass of k-uniformly convex functions order α type β with varying argument of coefficients and obtain coefficient estimates. Further we investigate extreme points, growth and distortion bounds, radii of starlikeness and convexity and modified Hadamard products.

MSC: 30C45

1. Introduction

Denoted by S the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

that are analytic and univalent in the unit disc $\mathcal{U} = \{z : |z| < 1\}$ and by S^* and $\mathcal{K}$ the subclasses of S that are respectively, starlike and convex. Goodman [1,2] introduced and defined the following subclasses of $\mathcal{K}$ and S^*. A function $f(z)$ is uniformly convex (uniformly starlike) in $\mathcal{U}$ if $f(z)$ is in $\mathcal{K}$ (S^*) and has the property that for every circular arc γ contained in $\mathcal{U}$, with center ξ also in $\mathcal{U}$, the arc $f(\gamma)$ is convex (starlike) with respect to $f(\xi)$. The class of uniformly convex functions denoted by UCV and the class of uniformly starlike functions by UST (for details see [1]). It is well known from [3,4] that

$$f \in UCV \iff \Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geqslant \left|\frac{zf''(z)}{f'(z)}\right|.$$

In [4], Rønning introduced a new class of starlike functions related to UCV and defined as

$$f \in S_p \iff \Re\left\{\frac{zf'(z)}{f(z)}\right\} \geqslant \left|\frac{zf'(z)}{f(z)} - 1\right|.$$

Note that $f(z) \in UCV \iff zf'(z) \in S_p$. Further Rønning generalized the class S_p by introducing a parameter α, $-1 \leqslant \alpha < 1$,

$$f \in S_p(\alpha) \iff \Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geqslant \left|\frac{zf'(z)}{f(z)} - 1\right|.$$

In 1997, Bharati et al. [5] introduced the following classes of k-starlike functions of order α (k-$ST(\alpha)$) and k-uniformly convex functions of order α (k-$UCV(\alpha)$).

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Peer review under responsibility of Egyptian Mathematical Society.

ELSEVIER Production and hosting by Elsevier

1110-256X
http://dx.doi.org/10.1016/j.joems.2013.02.005

$$f \in k - ST(\alpha) \Longleftrightarrow \Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geqslant k\left|\frac{zf'(z)}{f(z)} - 1\right|,$$
$$k \geqslant 0, \quad 0 \leqslant \alpha < 1,$$

and

$$f \in k - UCV(\alpha) \Longleftrightarrow \Re\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} \geqslant k\left|\frac{zf''(z)}{f'(z)}\right|,$$
$$k \geqslant 0, \quad 0 \leqslant \alpha < 1.$$

It follows that $f \in k - UCV(\alpha) \iff zf' \in k - ST(\alpha)$.

Recently, Sim et al. [6] introduced the subclasses $k - UCV(\alpha, \beta)$ and $k - ST(\alpha, \beta)$ of the univalent function class S as follows (see El-Ashwah et al. [7]):

$$f \in k - UCV(\alpha, \beta) \Longleftrightarrow \Re\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} \geqslant k\left|1 + \frac{zf''(z)}{f'(z)} - \beta\right|,$$

where $0 \leqslant \alpha < \beta \leqslant 1$ and $k(1-\beta) < (1-\alpha)$ and

$$f \in k - ST(\alpha, \beta) \Longleftrightarrow \Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geqslant k\left|\frac{zf'(z)}{f(z)} - \beta\right|,$$

where $0 \leqslant \alpha < \beta \leqslant 1$ and $k(1-\beta) < (1-\alpha)$. Notice that $f \in k\text{-}UCV(\alpha, \beta) \iff zf' \in k\text{-}ST(\alpha, \beta)$.

Motivated by the above works, we define a unified subclass of univalent function class S as follows:

$$\Re\left\{\frac{zf'(z) + (1+2\lambda)z^2f''(z) + \lambda z^3 f'''(z)}{zf'(z) + \lambda z^2 f''(z)} - \alpha\right\} > k\left|\frac{zf'(z) + (1+2\lambda)z^2f''(z) + \lambda z^3 f'''(z)}{zf'(z) + \lambda z^2 f''(z)} - \beta\right|, \quad z \in \mathcal{U}. \tag{1.2}$$

We also let $\mathcal{VU}_\eta(\lambda, \alpha, \beta, k) = \mathcal{U}(\lambda, \alpha, \beta, k) \bigcap V_\eta$, where $\mathcal{V}_\eta$ the class of functions $f \in S$ of the form (1.1) for which $arg(a_n) = \pi + (n-1)\eta$, $n \geqslant 2$. For $\eta = 0$, we obtain the familiar class T of functions with negative coefficients [8]. Moreover, we define $\mathcal{V} := \cup_{\eta \in \mathbb{R}} \mathcal{V}_\eta$. The class $\mathcal{V}$ was introduced by Sliverman [9] (see also [10]). It is called the class of functions with varying argument of coefficients. We note that, by specializing the parameters λ, α, and k we obtain the following subclasses studied by various authors.

(1) $\mathcal{VU}_\eta(0, \alpha, 1, 0) = \mathcal{VK}(\alpha)$ [9].
(2) $\mathcal{VU}_0(0, \alpha, 1, 0) = \mathcal{K}(\alpha)$ [8].
(3) $\mathcal{VU}_0(0, \alpha, 1, k) = k - UCV(\alpha)$ [5,11].
(4) $\mathcal{U}(0, \alpha, 1, 1) = UCV(\alpha)$ [4].
(5) $\mathcal{U}(0, 1, 1, k) = k - UCV$ [12].
(6) $\mathcal{U}(0, \alpha, \beta, k) = k - UCV(\alpha, \beta)$ [6].
(7) $\mathcal{VU}_\eta(0, \alpha, \beta, k) = k - UCV(\alpha, \beta)$ [7].
(8) $\mathcal{VU}_0(\lambda, \alpha, 1, 0) = \mathcal{U}(\lambda, \alpha)$ [13].
(9) $\mathcal{VU}_0(\lambda, \alpha, 1, k) = \mathcal{U}(\lambda, \alpha, k)$ [14].

The main object of this paper is to obtain a sufficient coefficient condition for functions f of the form (1.1) to be in the class $\mathcal{VU}_\eta(\lambda, \alpha, \beta, k)$ and we show that it is also a necessary condition for functions belong to this class. Further we investigate extreme points, growth and distortion bounds, radii of starlikeness and convexity and modified Hadamard products for the class $\mathcal{VU}_\eta(\lambda, \alpha, \beta, k)$.

2. Coefficient estimates

In this section we obtain a necessary and sufficient condition for functions $f(z)$ in the classes $\mathcal{VU}_\eta(\lambda, \alpha, \beta, k)$.

Theorem 2.1. *A function $f(z)$ of the form (1.1) is in $\mathcal{U}(\lambda, \alpha, \beta, k)$ if*

$$\sum_{n=2}^{\infty}[\varphi_n(1+k) - (k\beta + \alpha)\psi_n]|a_n| \leqslant 1 - \alpha - k(1-\beta), \tag{2.1}$$

where

$$\varphi_n = n^2[1 + \lambda(n-1)], \quad \psi_n = n[1 + \lambda(n-1)], \tag{2.2}$$

and $-1 \leqslant \alpha < \beta \leqslant 1$, $0 \leqslant \lambda < 1$, $k(1-\beta) < 1 - \alpha$ and $z \in \mathcal{U}$.

Proof. It suffices to show that the inequality (1.2) holds true. Upon using the fact that

$$\Re(w) > k|w - \beta| + \alpha \quad \text{iff} \quad \Re((1 + ke^{i\theta})w - \beta ke^{i\theta}) > \alpha, \tag{2.3}$$

then the inequality (1.2) may be written as

$$\Re\left((1 + ke^{i\theta})\frac{zf'(z) + (1+2\lambda)z^2 f''(z) + \lambda z^3 f'''(z)}{zf'(z) + \lambda z^2 f''(z)} - \beta ke^{i\theta}\right) \geqslant \alpha.$$

That is,

$$\Re\left(\frac{A(z)}{B(z)}\right) > \alpha,$$

where

$$A(z) = (1 + ke^{i\theta})[zf'(z) + (1+2\lambda)z^2 f''(z) + \lambda z^3 f'''(z)] - \beta ke^{i\theta}[zf'(z) + \lambda z^2 f''(z)],$$

and

$$B(z) = zf'(z) + \lambda z^2 f''(z),$$

then we have

$$|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| \geqslant 0. \tag{2.4}$$

Now,

$$\begin{aligned}|A(z)+(1-\alpha)B(z)| &= |((1-\beta)ke^{i\theta} + 2 - \alpha)z - \sum_{n=2}^{\infty}[(\beta\psi_n - \varphi_n)ke^{i\theta} - (1-\alpha)\psi_n - \varphi_n]a_n z^n| \\ &\geqslant (-(1-\beta)k + 2 - \alpha)|z| - \sum_{n=2}^{\infty}[(\beta\psi_n - \varphi_n)k + (1-\alpha)\psi_n + \varphi_n]|a_n||z|^n,\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}|A(z)-(1+\alpha)B(z)| &= |((1-\beta)ke^{i\theta} - \alpha)z + \sum_{n=2}^{\infty}[(\varphi_n - \beta\psi_n)ke^{i\theta} + \varphi_n - (1+\alpha)\psi_n]a_n z^n| \\ &\leqslant ((1-\beta)k + \alpha)|z| + \sum_{n=2}^{\infty}[(\varphi_n - \beta\psi_n)k - (1+\alpha)\psi_n + \varphi_n]|a_n||z|^n.\end{aligned} \tag{2.6}$$

From (2.5) and (2.6), we have

$$
\begin{aligned}
&|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)| \\
&\geqslant [2(1-\alpha)-2k(1-\beta)]|z|-2\sum_{n=2}^{\infty}[(\varphi_n-\beta\psi_n)k+(\varphi_n \\
&\quad -\alpha\psi_n)]|a_n||z|^n \\
&=2\left[[(1-\alpha)-k(1-\beta)]|z|-\sum_{n=2}^{\infty}[\varphi_n(1+k)-\psi_n(k\beta+\alpha)]|a_n||z|^n\right].
\end{aligned}
$$

The last expression is bounded below by 0 if

$$
\sum_{n=2}^{\infty}[\varphi_n(1+k)-(k\beta+\alpha)\psi_n]|a_n| \leqslant 1-\alpha-k(1-\beta),
$$

and hence the proof is complete. □

In the following theorem, it is shown that the condition (2.1) is also necessary for functions $f \in \mathcal{VU}_\eta(\lambda,\alpha,\beta,k)$.

Theorem 2.2. *Let $f(z)$ of the form (1.1) and in $\mathcal{V}_\eta$, then $f \in \mathcal{VU}_\eta(\lambda,\alpha,\beta,k)$ if and only if*

$$
\sum_{n=2}^{\infty}[\varphi_n(1+k)-(k\beta+\alpha)\psi_n]|a_n| \leqslant 1-\alpha-k(1-\beta), \tag{2.7}
$$

where φ_n and ψ_n are given by (2.2).

Proof. In view of Theorem 2.1, we need only to show that $f(z) \in \mathcal{VU}_\eta(\lambda,\alpha,\beta,k)$ satisfies the coefficient inequality (2.7). If $f(z) \in \mathcal{VU}_\eta(\lambda,\alpha,\beta,k)$ then by definition, we have

$$
\begin{aligned}
&\Re\left(\frac{(1-\alpha)+\sum_{n=2}^{\infty}(\varphi_n-\alpha\psi_n)a_nz^{n-1}}{1+\sum_{n=2}^{\infty}\psi_na_nz^{n-1}}\right) \\
&\quad \geqslant k\left|\frac{(1-\beta)+\sum_{n=2}^{\infty}(\varphi_n-\beta\psi_n)a_nz^{n-1}}{1+\sum_{n=2}^{\infty}\psi_na_nz^{n-1}}\right|.
\end{aligned}
$$

Since f is a function of the form (1.1) with the argument property given in the class $\mathcal{V}_\eta$ and setting $z=re^{i\eta}$ in the above inequality, we have

$$
\begin{aligned}
&\frac{(1-\alpha)-\sum_{n=2}^{\infty}(\varphi_n-\alpha\psi_n)|a_n|r^{n-1}}{1-\sum_{n=2}^{\infty}\psi_n|a_n|r^{n-1}} \\
&\quad \geqslant k\frac{(1-\beta)+\sum_{n=2}^{\infty}(\varphi_n-\beta\psi_n)|a_n|r^{n-1}}{1-\sum_{n=2}^{\infty}\psi_n|a_n|r^{n-1}}.
\end{aligned} \tag{2.8}
$$

Letting $r \to 1$, (2.8) leads the desired inequality

$$
\begin{aligned}
&\sum_{n=2}^{\infty}[\varphi_n(1+k)-(k\beta+\alpha)\psi_n]|a_n| \leqslant 1-\alpha-k(1-\beta), \\
&\quad -1 \leqslant \alpha < 1, \quad k \geqslant 0.
\end{aligned}
$$

Finally, the function $f(z)$ given by

$$
\begin{aligned}
&f_{n,\eta}(z)=z-\frac{[1-\alpha-k(1-\beta)]e^{i(1-n)\eta}}{[M_n(1+k)-(k\beta+\alpha)F_n]}z^n, \\
&\quad 0 \leqslant \eta \leqslant 2\pi, \quad n \geqslant 2,
\end{aligned} \tag{2.9}
$$

where φ_2 and ψ_2 as written in (2.2), is extremal for the function. □

Corollary 2.3. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{VU}_\eta(\lambda,\alpha,\beta,k)$. Then*

$$
|a_n| \leqslant \frac{1-\alpha-k(1-\beta)}{[\varphi_n(1+k)-(k\beta+\alpha)\psi_n]}, \; n \geqslant 2. \tag{2.10}
$$

The equality in (2.10) is attained for the function $f(z)$ given by (2.9).

3. Growth and distortion theorems

In this section we obtain growth and distortion bounds for functions in the class $\mathcal{VU}_\eta(\lambda,\alpha,\beta,k)$.

Theorem 3.1. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{VU}_\eta(\lambda,\alpha,\beta,k)$. Then for $|z| < r = 1$*

$$
\begin{aligned}
&r-\frac{1-\alpha-k(1-\beta)}{[\varphi_2(1+k)-(k\beta+\alpha)\psi_2]}r^2 \leqslant |f(z)| \\
&\quad \leqslant r+\frac{1-\alpha-k(1-\beta)}{[\varphi_2(1+k)-(k\beta+\alpha)\psi_2]}r^2.
\end{aligned} \tag{3.1}
$$

The result (3.1) is attained for the function $f(z)$ given by (2.9) for $z=\pm r$.

Proof. Note that

$$
\begin{aligned}
[\varphi_2(1+k)-(k\beta+\alpha)\psi_2]\sum_{n=2}^{\infty}|a_n| &\leqslant \sum_{n=2}^{\infty}[\varphi_n(1+k)-(k\beta \\
&\quad +\alpha)\psi_n]|a_n| \\
&\leqslant 1-\alpha-k(1-\beta),
\end{aligned}
$$

this last inequality follows from Theorem 2.2. Thus

$$
\begin{aligned}
|f(z)| &\geqslant |z|-\sum_{n=2}^{\infty}|a_n||z|^n \geqslant r-r^2\sum_{n=2}^{\infty}|a_n| \\
&\geqslant r-\frac{1-\alpha-k(1-\beta)}{[\varphi_2(1+k)-(k\beta+\alpha)\psi_2]}r^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|f(z)| &\leqslant |z|+\sum_{n=2}^{\infty}|a_n||z|^n \leqslant r+r^2\sum_{n=2}^{\infty}|a_n| \\
&\leqslant r+\frac{1-\alpha-k(1-\beta)}{[\varphi_2(1+k)-(k\beta+\alpha)\psi_2]}r^2.
\end{aligned}
$$

This completes the proof. □

Theorem 3.2. *Let the function $f(z)$ defined by (1.1) be in the class $\mathcal{VU}_\eta(\lambda,\alpha,\beta,k)$. Then for $|z| < r = 1$*

$$
\begin{aligned}
&1-\frac{2(1-\alpha-k(1-\beta))}{[\varphi_2(1+k)-(k\beta+\alpha)\psi_2]}r \leqslant |f'(z)| \\
&\quad \leqslant 1+\frac{2(1-\alpha-k(1-\beta))}{[\varphi_2(1+k)-(k\beta+\alpha)\psi_2]}r.
\end{aligned} \tag{3.2}
$$

Proof. We have

$$
|f'(z)| \geqslant 1-\sum_{n=2}^{\infty}n|a_n||z|^{n-1} \geqslant 1-r\sum_{n=2}^{\infty}n|a_n|, \tag{3.3}
$$

and

$$|f'(z)| \leqslant 1 + \sum_{n=2}^{\infty} n|a_n||z|^{n-1} \leqslant 1 + r\sum_{n=2}^{\infty} n|a_n|. \quad (3.4)$$

In view of Theorem 2.2,

$$\frac{[\varphi_2(1+k) - (k\beta+\alpha)\psi_2]}{2}\sum_{n=2}^{\infty} n|a_n| \leqslant \sum_{n=2}^{\infty}[\varphi_n(1+k) - (k\beta+\alpha)\psi_n]|a_n| \leqslant 1-\alpha-k(1-\beta), \quad (3.5)$$

or, equivalently

$$\sum_{n=2}^{\infty} n|a_n| \leqslant \frac{2(1-\alpha-k(1-\beta))}{[\varphi_2(1+k) - (k\beta+\alpha)\psi_2]}. \quad (3.6)$$

A substitution of (3.6) into (3.3) and (3.4) yields the inequality (3.2). This completes the proof. □

Theorem 3.3. *Let $f \in \mathcal{VU}_\eta(\lambda, \alpha, \beta, k)$ with argument property as in the class $\mathcal{V}_\eta$. Define $f_j(z) = z$, and*

$$f_{n,\eta}(z) = z - \frac{[1-\alpha-k(1-\beta)]e^{i(1-n)\eta}}{[\varphi_n(1+k) - (k\beta+\alpha)\psi_n]}z^n, \quad 0 \leqslant \eta \leqslant 2\pi, \quad n \geqslant 2. \quad (3.7)$$

Then $f(z)$ is in the class $\mathcal{VU}_\eta(\lambda, \alpha, \beta, k)$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=1}^{\infty}\mu_n f_{n,\eta}(z), \quad (3.8)$$

where $\mu_n \geqslant 0 (n \geqslant 1)$ and $\sum_{n=1}^{\infty}\mu_n = 1$.

Proof. Assume that

$$f(z) = \mu_1 f_1(z) + \sum_{n=2}^{\infty}\mu_n\left[z - \frac{[1-\alpha-k(1-\beta)]e^{i(1-n)\eta}}{[\varphi_n(1+k) - (k\beta+\alpha)\psi_n]}z^n\right] = \sum_{n=1}^{\infty}\mu_n z - \sum_{n=2}^{\infty}\frac{[1-\alpha-k(1-\beta)]e^{i(1-n)\eta}}{[\varphi_n(1+k) - (k\beta+\alpha)\psi_n]}\mu_n z^n.$$

Then it follows that

$$\sum_{n=2}^{\infty}\left|\frac{[1-\alpha-k(1-\beta)]e^{i(1-n)\eta}}{[\varphi_n(1+k) - (k\beta+\alpha)\psi_n]}\mu_n\right|[\varphi_n(1+k) - (k\beta+\alpha)\psi_n] = \sum_{n=2}^{\infty}\mu_n[1-\alpha-k(1-\beta)] \leqslant (1-\mu_1)[1-\alpha-k(1-\beta)] \leqslant 1-\alpha-k(1-\beta),$$

so by Theorem 2.2, $f(z) \in \mathcal{VU}_\eta(\lambda, \alpha, \beta, k)$. Conversely, assume that the function $f(z)$ defined by (1.1) belongs to the class $\mathcal{VU}_\eta(\lambda, \alpha, \beta, k)$, then

$$|a_n| \leqslant \frac{1-\alpha-k(1-\beta)}{[\varphi_n(1+k) - (k\beta+\alpha)\psi_n]}, \quad n \geqslant 2.$$

Setting $\mu_n = \frac{[\varphi_n(1+k)-(k\beta+\alpha)\psi_n]}{1-\alpha-k(1-\beta)}|a_n|, (n \geqslant 2)$ and $\mu_1 = 1 - \sum_{n=2}^{\infty}\mu_n$, $n \geqslant 2$. Then $f(z) = \sum_{n=1}^{\infty}\mu_n f_{n,\eta}(z)$ and this completes the proof. □

4. Radii of close-to-convexity, starlikeness and convexity

In this section, we obtain the radii of close-to-convexity, starlikeness and convexity for the class $\mathcal{VU}_\eta(\lambda, \alpha, \beta, k)$.

Theorem 4.1. *Let $f \in \mathcal{VU}_\eta(\lambda, \alpha, \beta, k)$. Then $f(z)$ is close-to-convex of order $\sigma (0 \leqslant \sigma < 1)$ in the disc $|z| < r_1$, where*

$$r_1 := \inf\left[\frac{(1-\sigma)[\varphi_n(1+k) - (k\beta+\alpha)\psi_n]}{n(1-\alpha-k(1-\beta))}\right]^{\frac{1}{n-1}}, \quad n \geqslant 2. \quad (4.1)$$

The result is sharp, with extremal function $f(z)$ given by (2.9).

Proof. Given $f \in \mathcal{V}_\eta$, and f is close-to-convex of order σ, we have

$$|f'(z) - 1| < 1 - \sigma. \quad (4.2)$$

For the left hand side of (4.2) we have

$$|f'(z) - 1| \leqslant \sum_{n=2}^{\infty} n|a_n||z|^{n-1}.$$

The last expression is less than $1-\sigma$ if

$$\sum_{n=2}^{\infty}\frac{n}{1-\sigma}|a_n||z|^{n-1} < 1.$$

Using the fact, that $f \in \mathcal{VU}_\eta(\lambda, \alpha, \beta, k)$, if and only if

$$\sum_{n=2}^{\infty}\frac{[\varphi_n(1+k) - (k\beta+\alpha)\psi_n]}{(1-\alpha-k(1-\beta))}|a_n| \leqslant 1.$$

We can say (4.2) is true if

$$\frac{n}{1-\sigma}|z|^{n-1} \leqslant \frac{[\varphi_n(1+k) - (k\beta+\alpha)\psi_n]}{(1-\alpha-k(1-\beta))}.$$

Or, equivalently,

$$|z|^{n-1} = \left[\frac{(1-\sigma)[\varphi_n(1+k) - (k\beta+\alpha)\psi_n]}{n(1-\alpha-k(1-\beta))}\right],$$

which completes the proof. □

Theorem 4.2. *Let $f \in \mathcal{VU}_\eta(\lambda, \alpha, \beta, k)$. Then*

(i) *f is starlike of order $\sigma (0 \leqslant \sigma < 1)$ in the disc $|z| < r_2$; where*

$$r_2 = \inf\left[\left(\frac{1-\sigma}{n-\sigma}\right)\frac{[\varphi_n(1+k) - (k\beta+\alpha)\psi_n]}{(1-\alpha-k(1-\beta))}\right]^{\frac{1}{n-1}}, \quad n \geqslant 2, \quad (4.3)$$

(ii) *f is convex of order σ $(0 \leqslant \sigma < 1)$ in the unit disc $|z| < r_3$, where*

$$r_3 = \inf\left[\left(\frac{1-\sigma}{n(n-\sigma)}\right)\frac{[\varphi_n(1+k) - (k\beta+\alpha)\psi_n]}{(1-\alpha-k(1-\beta))}\right]^{\frac{1}{n-1}}, \quad n \geqslant 2. \quad (4.4)$$

Each of these results are sharp for the extremal function $f(z)$ given by (2.9).

Proof.

(i) Given $f \in \mathcal{V}_\eta$, and f is starlike of order σ, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \sigma. \tag{4.5}$$

For the left hand side of (4.5) we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leqslant \frac{\sum_{n=2}^{\infty}(n-1)|a_n||z|^{n-1}}{1 - \sum_{n=2}^{\infty}|a_n||z|^{n-1}}.$$

The last expression is less than $1 - \sigma$ if

$$\sum_{n=2}^{\infty}\frac{n-\sigma}{1-\sigma}|a_n||z|^{n-1} < 1.$$

Using the fact, that $f \in \mathcal{VU}_\eta(\lambda, \alpha, \beta, k)$ if and only if

$$\sum_{n=2}^{\infty}\frac{[\varphi_n(1+k) - (k\beta+\alpha)\psi_n]}{(1-\alpha-k(1-\beta))}|a_n| \leqslant 1.$$

We can say (4.5) is true if

$$\frac{n-\sigma}{1-\sigma}|z|^{n-1} < \frac{[\varphi_n(1+k) - (k\beta+\alpha)\psi_n]}{(1-\alpha-k(1-\beta))}.$$

Or, equivalently,

$$|z|^{n-1} = \left[\left(\frac{1-\sigma}{n-\sigma}\right)\frac{[\varphi_n(1+k) - (k\beta+\alpha)\psi_n]}{(1-\alpha-k(1-\beta))}\right]$$

which yields the starlikeness of the family.

(ii) Using the fact that f is convex if and only if zf' is starlike, we can prove (ii), on lines similar to the proof of (i). □

5. Results on modified hadamard product

Let the functions $f_j(z)$ $(j = 1, 2)$ be defined by

$$f_j(z) = z + \sum_{n=2}^{\infty}a_{n,i}z^n, \qquad a_{n,i} \geqslant 0; \quad i \in \mathbb{N}, \tag{5.1}$$

then we define the modified Hadamard product of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = z - \sum_{n=2}^{\infty}a_{n,1}a_{n,2}z^n. \tag{5.2}$$

Now, we prove the following.

Theorem 5.1. *Let each of the functions $f_j(z)$ $(j = 1, 2)$ defined by (5.1) be in the class $\mathcal{VU}_\eta(\lambda, \alpha, \beta, k)$. Then $(f_1 * f_2) \in \mathcal{VU}_\eta(\lambda, \delta_1, k)$, for*

$$\delta_1 = \frac{[\varphi_n(1+k) - (k\beta+\alpha)\psi_n]^2 - [\varphi_n(1+k) - k\beta\psi_n](1-\alpha-k(1-\beta))^2}{[\varphi_n(1+k) - (k\beta+\alpha)\psi_n]^2 - \psi_n(1-\alpha-k(1-\beta))^2}. \tag{5.3}$$

The result is sharp.

Proof. We need to prove the largest δ_1 such that

$$\sum_{n=2}^{\infty}\frac{[\varphi_n(1+k) - (\delta_1+k\beta)\psi_n]}{(1-\delta_1-k(1-\beta))}a_{n,1}a_{n,2} \leqslant 1. \tag{5.4}$$

From Theorem 2.2, we have

$$\sum_{n=2}^{\infty}\frac{[\varphi_n(1+k) - (k\beta+\alpha)\psi_n]}{1-\alpha-k(1-\beta)}a_{n,1} \leqslant 1,$$

and

$$\sum_{n=2}^{\infty}\frac{[\varphi_n(1+k) - (k\beta+\alpha)\psi_n]}{1-\alpha-k(1-\beta)}a_{n,2} \leqslant 1,$$

by the Cauchy–Schwarz inequality, we have

$$\sum_{n=2}^{\infty}\frac{[\varphi_n(1+k) - (k\beta+\alpha)\psi_n]}{1-\alpha-k(1-\beta)}\sqrt{a_{n,1}a_{n,2}} \leqslant 1. \tag{5.5}$$

Thus it is sufficient to show that

$$\frac{[\varphi_n(1+k) - (\delta_1+k\beta)\psi_n]}{(1-\delta_1-k(1-\beta))}a_{n,1}a_{n,2} \leqslant \frac{[\varphi_n(1+k) - (k\beta+\alpha)\psi_n]}{1-\alpha-k(1-\beta)}\sqrt{a_{n,1}a_{n,2}}, \quad n \geqslant 2, \tag{5.6}$$

that is, for n $\geqslant 2$

$$\sqrt{a_{n,1}a_{n,2}} \leqslant \frac{[\varphi_n(1+k) - (k\beta+\alpha)\psi_n](1-\delta_1-k(1-\beta))}{[\varphi_n(1+k) - (\delta_1+k\beta)\psi_n](1-\alpha-k(1-\beta))}, \quad n \geqslant 2. \tag{5.7}$$

Note that

$$\sqrt{a_{n,1}a_{n,2}} \leqslant \frac{(1-\alpha-k(1-\beta))}{[\varphi_n(1+k) - (k\beta+\alpha)\psi_n]}, \quad n \geqslant 2. \tag{5.8}$$

Consequently, we need only to prove that

$$\frac{(1-\alpha-k(1-\beta))}{[\varphi_n(1+k) - (k\beta+\alpha)\psi_n]} \leqslant \frac{[\varphi_n(1+k) - (k\beta+\alpha)\psi_n](1-\delta_1-k(1-\beta))}{[\varphi_n(1+k) - (\delta_1+k\beta)\psi_n](1-\alpha-k(1-\beta))}, \tag{5.9}$$

or equivalently

$$\delta_1 \leqslant \frac{[\varphi_n(1+k) - (k\beta+\alpha)\psi_n]^2 - [\varphi_n(1+k) - k\beta\psi_n](1-\alpha-k(1-\beta))^2}{[\varphi_n(1+k) - (k\beta+\alpha)\psi_n]^2 - \psi_n(1-\alpha-k(1-\beta))^2} = \Delta(n). \tag{5.10}$$

Since $\Delta(n)$ is an increasing function of $n (n \geqslant 2)$, letting $n = 2$ in (5.10) we obtain

$$\delta_1 \leqslant \Delta(2) = \frac{[\varphi_2(1+k) - (\alpha+k\beta)\psi_2]^2 - [\varphi_2(1+k) - k\beta\psi_2](1-\alpha-k(1-\beta))^2}{[\varphi_2(1+k) - (\alpha+k\beta)\psi_2]^2 - \psi_2(1-\alpha-k(1-\beta))^2}, \tag{5.11}$$

which proves the main assertion of Theorem 5.1. The result is sharp for the functions defined by (2.9). □

Theorem 5.2. *Let the function $f_j(z)$ $(j = 1,2)$ defined by (5.1) be in the class $\mathcal{VU}_\eta(\lambda, \alpha, \beta, k)$. If the sequence $\{\varphi_n(1+k) - (k\beta+\alpha)\psi_n\}$ is non-decreasing. Then the function*

$$h(z) = z - \sum_{n=2}^{\infty}\left(a_{n,1}^2 + a_{n,2}^2\right)z^n \tag{5.12}$$

belongs to the class $\mathcal{VU}_\eta(\lambda, \delta_2, k)$ where

$$\delta_2 = \frac{[\varphi_n(1+k)-(k\beta+\alpha)\psi_n]^2 - 2[\varphi_n(1+k)-k\beta\psi_n](1-\alpha-k(1-\beta))^2}{[\varphi_n(1+k)-(k\beta+\alpha)\psi_n]^2 - 2\psi_n(1-\alpha-k(1-\beta))^2}.$$

Proof. By virtue of Theorem 2.2, we have for $f_j(z)\ (j=1,2) \in \mathcal{VU}_\eta(\lambda,\alpha,\beta,k)$ we have

$$\sum_{n=2}^{\infty}\left[\frac{[\varphi_n(1+k)-(k\beta+\alpha)\psi_n]}{1-\alpha-k(1-\beta)}\right]^2 a_{n,1}^2 \leqslant \sum_{n=2}^{\infty}\left[\frac{[\varphi_n(1+k)-(k\beta+\alpha)\psi_n]}{1-\alpha-k(1-\beta)}a_{n,1}\right]^2 \leqslant 1, \tag{5.13}$$

and

$$\sum_{n=2}^{\infty}\left[\frac{[\varphi_n(1+k)-(k\beta+\alpha)\psi_n]}{1-\alpha-k(1-\beta)}\right]^2 a_{n,2}^2 \leqslant \sum_{n=2}^{\infty}\left[\frac{[\varphi_n(1+k)-(k\beta+\alpha)\psi_n]}{1-\alpha-k(1-\beta)}a_{n,2}\right]^2 \leqslant 1. \tag{5.14}$$

It follows from (5.13) and (5.14) that

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\frac{[\varphi_n(1+k)-(k\beta+\alpha)\psi_n]}{1-\alpha-k(1-\beta)}\right]^2\left(a_{n,1}^2+a_{n,2}^2\right) \leqslant 1. \tag{5.15}$$

Therefore we need to find the largest δ_2, such that

$$\frac{[\varphi_n(1+k)-(\delta_2+k\beta)\psi_n]}{(1-\delta_2-k(1-\beta))} \leqslant \frac{1}{2}\left[\frac{[\varphi_n(1+k)-(k\beta+\alpha)\psi_n]}{1-\alpha-k(1-\beta)}\right]^2, \quad n \geqslant 2$$

that is

$$\delta_2 \leqslant \frac{[\varphi_n(1+k)-(k\beta+\alpha)\psi_n]^2 - 2[\varphi_n(1+k)-k\beta\psi_n](1-\alpha-k(1-\beta))^2}{[\varphi_n(1+k)-(k\beta+\alpha)\psi_n]^2 - 2\psi_n(1-\alpha-k(1-\beta))^2} = \Psi(n).$$

Since $\Psi(n)$ is an increasing function of n, $(n \geqslant 2)$, we readily have

$$\delta_2 \leqslant \Psi(2) = \frac{[\varphi_2(1+k)-(\alpha+k\beta)\psi_2]^2 - 2[\varphi_2(1+k)-k\psi_2](1-\alpha-k(1-\beta))^2}{[\varphi_2(1+k)-(\alpha+k)\psi_2]^2 - 2\psi_2(1-\alpha-k(1-\beta))^2},$$

which completes the proof. □

Acknowledgements

The author would like to thank the referee for his valuable comments.

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